Chapter 3 – Exponential and Logarithmic Functions

<table>
<thead>
<tr>
<th>Section 1</th>
<th>Exponential Functions and Their Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 2</td>
<td>Logarithmic Functions and Their Graphs</td>
</tr>
<tr>
<td>Section 3</td>
<td>Properties of Logarithms</td>
</tr>
<tr>
<td>Section 4</td>
<td>Solving Exponential and Logarithmic Equations</td>
</tr>
<tr>
<td>Section 5</td>
<td>Exponential and Logarithmic Models</td>
</tr>
</tbody>
</table>

Vocabulary

- Exponential function
- Natural Base $e$
- Common Logarithmic Function
- Natural Logarithmic Function
- Change-of-base formula
Section 3.1 Exponential Functions and Their Graphs

Objective: In this lesson you learned how to recognize, evaluate, and graph exponential functions.

<table>
<thead>
<tr>
<th>Important Vocabulary</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Exponential Function</strong></td>
<td><strong>Natural Base e</strong></td>
</tr>
</tbody>
</table>

I. **Exponential Functions**

Polynomial functions and rational functions are examples of ____________________________ functions.

The exponential function \( f \) with base \( a \) is denoted by ____________________________, where \( a \geq 0, a \neq 1 \), and \( x \) is any real number.

II. **Graphs of Exponential Functions**

For \( a > 1 \), is the graph of \( f(x) = a^x \) increasing or decreasing over its domain? ____________________________

For \( a > 1 \), is the graph of \( g(x) = a^{-x} \) increasing or decreasing over its domain? ____________________________

For the graph of \( y = a^x \) or \( y = a^{-x}, a > 1 \), the domain is _________________, the range is _________________, and the y-intercept is ________________. Also, both graphs have ___________________ as a horizontal asymptote.

III. **The Natural Base \( e \)**

The natural exponential function is given by the function _______________.

For the graph of \( f(x) = e^x \), the domain is ________________, the range is ________________, and the y-intercept is ________________.

The number \( e \) can be approximated by the expression ________________ for large values of \( x \).
What you should learn:
How to recognize, evaluate, and graph exponential functions with base e

IV. Applications
After $t$ years, the balance $A$ in an account with principal $P$ and annual interest rate $r$ (in decimal form) is given by the formulas:

For $n$ compounding’s per year: ______________________

For continuous compounding: ______________________
Section 3.1 Examples – Exponential Functions and Their Graphs

(3) Using what you know from Chapter 1 (horizontal/vertical shifts, reflections, etc), describe the transformation from the graph of \( f \) to the graph of \( g \).

\[
f(x) = 3^x \quad g(x) = 3^{x-5}
\]

(4) Sketch a graph of the function by finding the asymptote(s) and calculating a few other points. State the domain and range in interval notation.

\[f(x) = 3^{x-1}\]

(5) Sketch a graph of the function by finding the asymptotes and calculating a few other points. State the domain and range in interval notation.

\[f(x) = 2 + e^{x-2}\]
Section 3.2 Logarithmic Functions and Their Graphs

Objective: In this lesson you learned how to recognize, evaluate, and graph logarithmic functions.

I. Logarithmic Functions
The logarithmic function with base \( a \) is the \( \text{_____________________________}\) of the exponential function \( f(x) = a^x \).

The logarithmic function with base \( a \) is defined as \( \text{_____________________________} \), for \( x > 0, a > 0 \), and \( a \neq 1 \), if and only if \( x = a^y \). The notation "\( \log_a x \)" is read as "\( \text{_____________________________} \)."

The equation \( x = a^y \) in exponential form is equivalent to the equation \( \text{_____________________________} \) in logarithmic form.

When evaluating logarithms, remember that a logarithm is a(n) \( \text{_____________________________} \). This means that \( \log_a x \) is the \( \text{_____________________________} \) to which \( a \) must be raised to obtain \( \text{___________} \).

Complete the following logarithm properties:

1) \( \log_a 1 = \text{___________} \)
2) \( \log_a a = \text{___________} \)
3) \( \log_a a^x = \text{___________} \)
4) \( a^{\log_a x} = \text{___________} \)
5) \[ \text{If } \log_a x = \log_a y, \text{ then } \text{___________} \]

Important Vocabulary

<table>
<thead>
<tr>
<th>Common Logarithmic Function</th>
<th>Natural Logarithmic Function</th>
</tr>
</thead>
</table>

What you should learn:
How to recognize and evaluate logarithmic functions with base \( a \)
II. Graphs of Logarithmic Functions
For $a > 1$, is the graph of $f(x) = \log_a x$ increasing or decreasing over its domain?

For the graph of $f(x) = \log_a x, a > 1$, the domain is _____________, the range is _____________, and the x-intercept is _______________.

Also, the graph has __________________________ as a vertical asymptote. The graph of $f(x) = \log_a x$ is a reflection of the graph of $f(x) = a^x$ over the line __________.

III. The Natural Logarithmic Function
Complete the following natural logarithm properties:

1) $\ln 1 = \underline{\phantom{0}}$
2) $\ln e = \underline{\phantom{0}}$
3) $\ln e^x = \underline{\phantom{0}}$
4) $e^{\ln x} = \underline{\phantom{0}}$
5) If $\ln x = \ln y$, then ______________.
Section 3.2 Examples – Logarithmic Functions and Their Graphs

(1) Write the logarithmic equation in exponential form.
   (a) $\log_4 64 = 3$  
   (b) $\log_5 \sqrt[3]{25} = \frac{2}{3}$

(2) Using what you know from Chapter 1 (horizontal/vertical shifts, reflections, etc), describe the transformation from the graph of $f$ to the graph of $g$.
   
   $f(x) = \log_2 x$  
   $g(x) = -2 + \log_2(x + 3)$

(3) Sketch a graph of the function by finding the asymptote(s) and calculating a few other points. State the domain and range in interval notation.
   
   $f(x) = \ln(x + 1)$

[Graph of $f(x) = \ln(x + 1)$]
Section 3.3 Properties of Logarithms

Objective: In this lesson you learned how to rewrite logarithmic functions with different bases and how to use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions.

### Important Vocabulary

**Change-of-Base Formula**

### I. Change of Base

Let \( a, b, \) and \( x \) be positive real numbers such that \( a \neq 1 \) and \( b \neq 1 \). The **change-of-base formula** states that:

### II. Properties of Logarithms

Let \( a \) be a positive number such that \( a \neq 1 \); let \( n \) be a real number; and let \( u \) and \( v \) be positive real numbers.

Complete the following logarithm properties:

1) \( \log_a(uv) = \text{________________________} \)

2) \( \log_a \frac{u}{v} = \text{________________________} \)

3) \( \log_a u^n = \text{________________________} \)

### III. Rewriting Logarithmic Expressions

To expand a logarithmic expression means to:

To condense a logarithmic expression means to:

Explained how to use a calculator to evaluate \( \log_b 20 \).
IV. Applications of Properties of Logarithms
One way of finding a model for a set of nonlinear data is to take
the natural log of each of the $x$-values and $y$-values of the data
set. If the points are graphed and fall on a straight line, then the
$x$-values and $y$-values are related by the equation _____________________________, where $m$ is
the slope of the straight line.
Section 3.3 Examples – Properties of Logarithms

(1) Rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.
\[ \log_5 x \]

(2) Use the properties of logarithms to rewrite and simplify the logarithmic expression.
\[ \log_2 4^2 \cdot 3^4 \]

(3) Use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms.
\[ \ln \frac{xy}{z} \]

(4) Condense the expression to the logarithm of a single quantity.
\[ 3 \log x + 2 \log y - 4 \log z \]
Section 3.4 Solving Exponential and Logarithmic Equations

Objective In the lesson you learned how to solve exponential and logarithmic equations.

I. Introduction
State the One-to-One Property for exponential equations.
State the One-to-One Property for logarithmic equations.
State the Inverse Property for exponential equations and for logarithmic equations.

Describe some strategies for using the One-to-One Properties and the Inverse Properties to solve exponential and logarithmic equations.

II. Solving Exponential Equations
Describe how to solve the exponential equation $10^x = 90$ algebraically.
III. Solving Logarithmic Equations
Describe how to solve the logarithmic equation $\log_6(4x - 7) = \log_6(8 - x)$ algebraically.

IV. Applications of Solving Exponential and Logarithmic Equations
Use the formula for continuous compounding $A = Pe^{rt}$, to find out how long it will take $1500$ to triple in value if it is invested at 12% interest, compounded continuously.
Section 3.4 Examples – Solving Exponential and Logarithmic Equations

(1) Solve the exponential equation.

\[ 5^x = \frac{1}{625} \]

(2) Solve the logarithmic equation.

\[ \ln(2x - 1) = 5 \]

(3) Solve the equation. Round your answer to three decimal places.

(a) \[ 7 - 2e^x = 5 \]  
(b) \[ \log x^2 = 6 \]
Section 3.5 Exponential and Logarithmic Models

Objective: In this lesson you learned how to use exponential growth models, exponential decay models, logistic models, and logarithmic models to solve real-life problems.

I. Introduction
   The *exponential growth model* is ________________.
   The *exponential decay model* is ________________.
   The *Gaussian model* is ________________.
   The *logistic growth model* is ________________.
   Logarithmic models are _________________ and ________________.

II. Exponential Growth and Decay
    To estimate the age of dead organic matter, scientists use the carbon dating model ________________, which denotes the ratio $R$ of carbon 14 to carbon 12 present at any time $t$ (in years).

III. Gaussian Models
    The Gaussian model is commonly used in probability and statistics to represent populations that are ________________.
    On a bell-shaped curve, the average value for a population is where the ________________ of the function occurs.

IV. Logarithmic Models
    The number of kitchen widgets $y$ (in millions) demanded each year is given by the model $y = 2 + 3 \ln(x + 1)$, where $x = 0$ represents the year 2000 and $x \geq 0$. Find the year in which the number of kitchen widgets demanded will be 8.6 million.

What you should learn:
- How to recognize the five most common types of models involving exponential or logarithmic functions
- How to use exponential growth and decay functions to model and solve real-life problems
- How to use Gaussian functions to model and solve real-life problems
- How to use logarithmic functions to model and solve real-life problems